Intro. to Financial Modeling

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Morgan Stanley

Stochastic Processes: Introduction

Overview

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- Stochastic Processes
  - Introduction
- Specific Processes
  - Random Walk
  - Arithmetic Brownian Motion
  - Geometric Brownian Motion
  - Mean Reverting Process (or Ornstein-Uhlenbeck Process)

Introduction

- We have regularly spoken of random variables as draws from an urn whose contents represent a certain distribution.
- We have worked with individual variables in Monte-Carlo simulation.
- We now want to simulate, not a one-time variable, but a series of variables that represents a process that goes through time and has some random component.

Introduction (cont'd)

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- To model any variable over time, we need an algorithm or formula that tells us how the variable changes from one period to the next.
- We simulate the process by applying the formula to an initial value to get the second value, applying it to the second value to get the third, etc.

Introduction (cont'd)

- Start with a deterministic process:
  - 0, 2, 4, 6, 8,…
  - The deterministic process is to add the value of 2 to the previous value.
  - More formally, we could describe this algorithm as:

$$X_{t+1} = X_t + 2, \quad X_0 = 0$$

Introduction (cont'd)

- Alternately, we could write the formula, not in terms of the new value of X, but in terms of the change in the value of X (ΔX):

$$\Delta X = 2\,\Delta t$$

- OK, uninteresting, but only the beginning. I will assume that you don't need the graph.

Introduction (cont'd)

- Stochastic Processes
  - These are similar to deterministic process, except that they add a chance element to each change.
    - The chance element is a random draw from a specified distribution. In our case this will be a normal distribution.

Introduction (cont'd)

- A simple example:
  - Flip a coin.
  - If heads, add 1.
  - If tails, subtract 1.
- Here are the results from my home experiment:

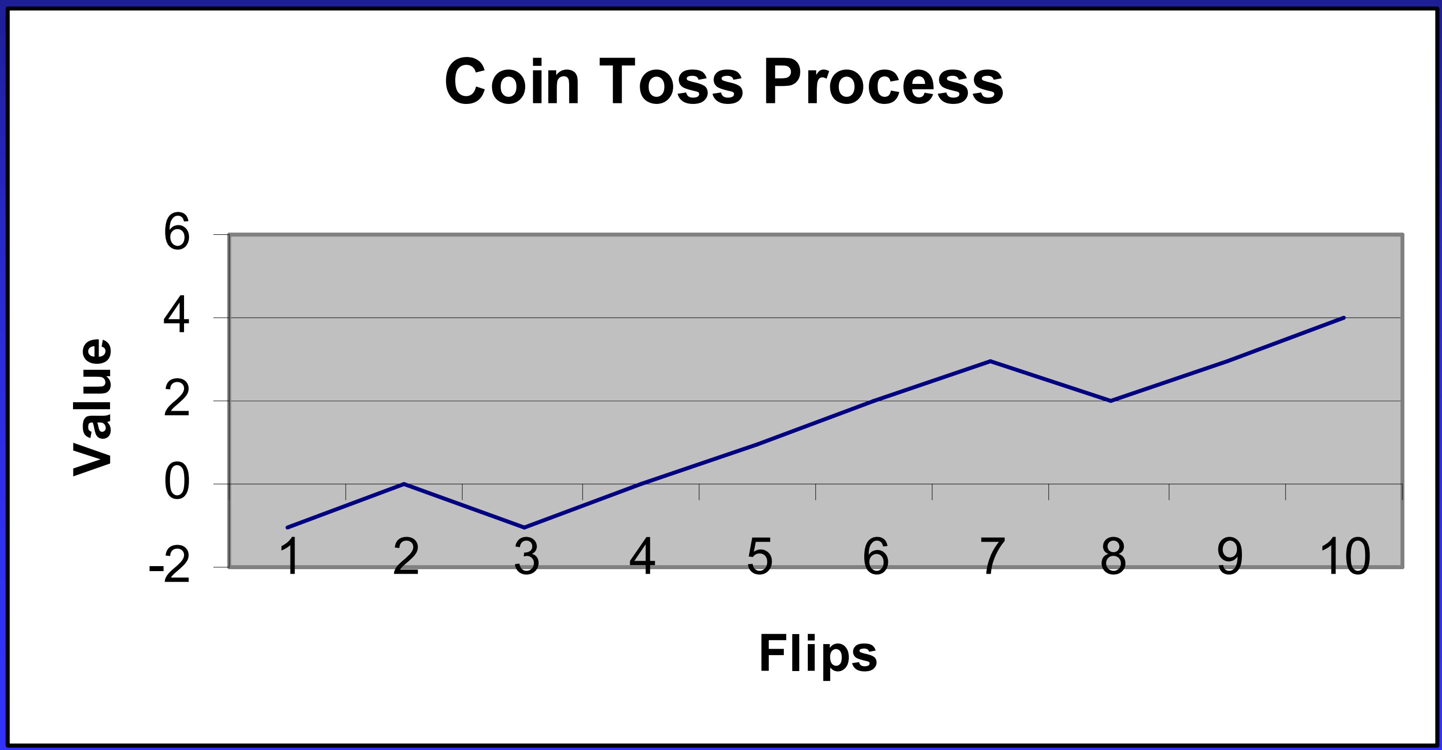
  T, H, T, H, H, H, H, T, H, H

  which produces

  -1, 0, -1, 0, 1, 2, 3, 2, 3, 4

Introduction (cont'd)

- As a graph



Introduction (cont'd)

- This is a stochastic process.
  - In fact, it is a very simple form of a 'random walk'.
  - Note that every time we do the experiment that the numbers come out different, **but** the statistical characteristics are constant.
  - What, on average, would you expect your find wealth to be?
- If we want to model an economic or financial variable, we need to find some process with similar characteristics.

Specific Stochastic Processes

Random Walk

Description

- In each period, there is either an increase of decrease that is independently and identically distributed (i.i.d.) and normally distributed with a mean of 0 and a variance of 1.
- In less politically correct times, this was described as the way that a quite inebriated walker proceeded.
- Experiment: ‘White Noise’

Random Walk

Uses

- Value of heads and tails in a random coin toss.
- The univariate position of a particle in physics.
- Used in early stock market models.

Random Walk

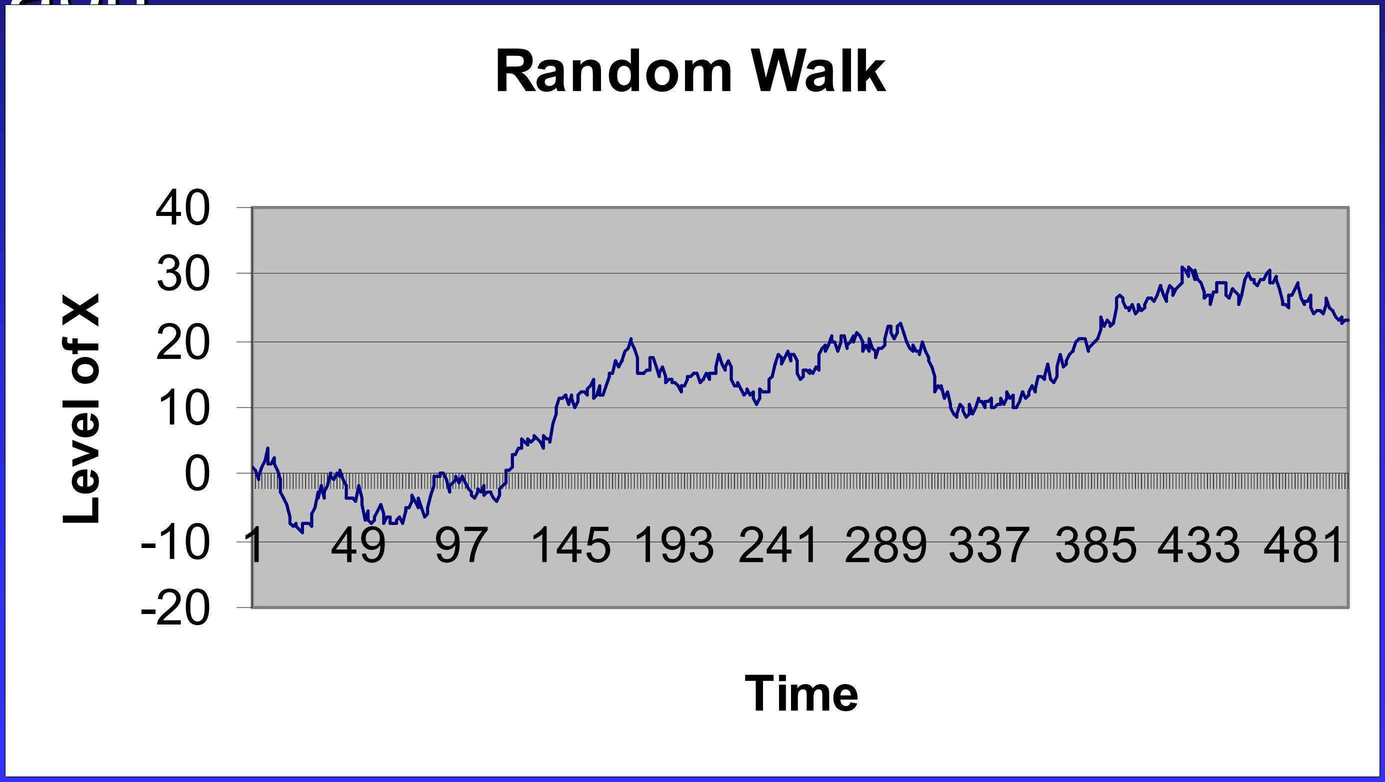
Formula

$$W_{t+1} = W_t + e_{t+1}$$

$$e \sim i.i.d.\, N(0,1)$$

Random Walk

Graph



Random Walk

Characteristics

- Range: -∞ to ∞

Arithmetic Brownian Motion

Description

- In each period, the change is a function of a constant and a random factor.
- Despite the random factor, the value increases in a generally linear fashion.
  - NOTE: since these are technically continuous, not discrete, processes, we use ‘d’, not ‘Δ’, for change, e.g., dX, not ΔX.

Arithmetic Brownian Motion

Uses

- Any variable that grows at a linear rate and shows increasing uncertainty.

Arithmetic Brownian Motion

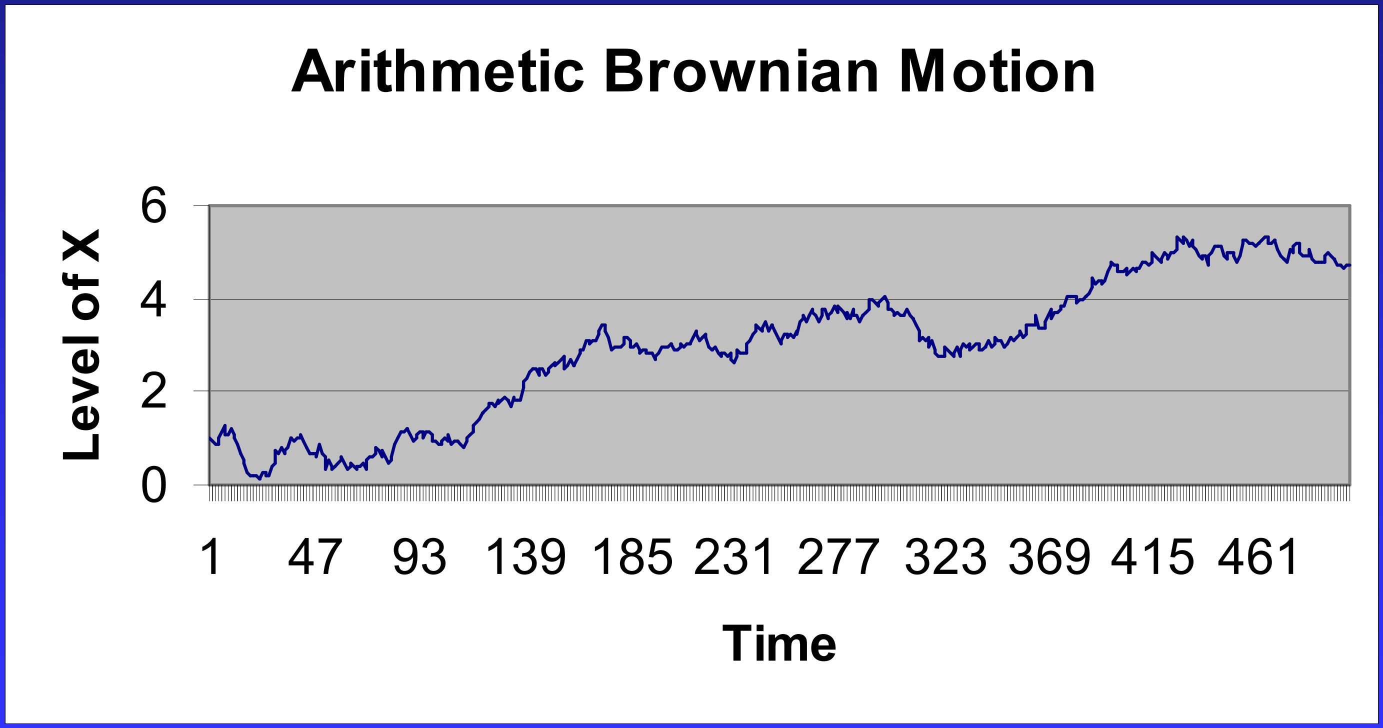
Formula

$$dX = \alpha\, dt + \sigma\, dW$$

- X = the random variable we are modeling
- dX = change in variable X
- α = drift
- dt = change in time
- σ = volatility
- dW = Weiner process

Arithmetic Brownian Motion

Graph



Arithmetic Brownian Motion

Characteristics

- X may be positive or negative
- Variance grows to infinity.

Geometric Brownian Motion

Description

- In each period, the change is a function of a constant times the variable and a random factor times the variable.
- The value increases in a generally geometric fashion.

Geometric Brownian Motion

Uses

- Any variable that grows exponentially with volatility proportional to the level of the variable.
- Stocks

Geometric Brownian Motion

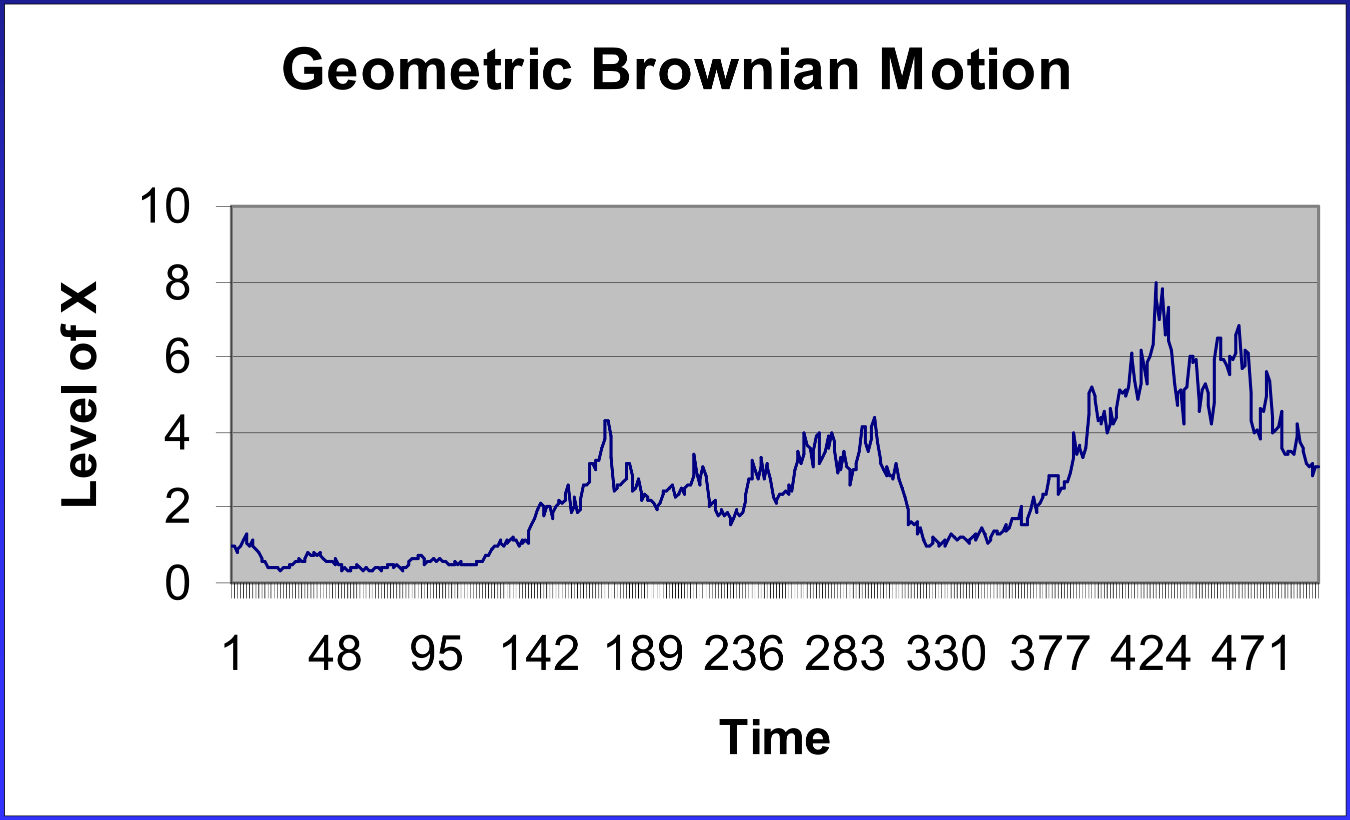
Formula

$$dX = \alpha X\, dt + \sigma X\, dW$$

- dX = change in variable X
- α = drift
- dt = change in time
- σ = volatility
- dW = Weiner process

Geometric Brownian Motion

Graph



Geometric Brownian Motion

Characteristics

- Exponential growth with an average rate of α.
- Volatility proportional to the level of the variable.
- If X begins at a positive value, it remains positive.
- Absorbing barrier at 0.

Geometric Brownian Motion

- Contrast
  - Geometric Brownian Motion with

$$dX = \alpha X\, dt + \sigma X\, dW$$

  - Arithmetic Brownian Motion

$$dX = \alpha\, dt + \sigma\, dW$$

Mean Reverting Process

Description

- This process ranges around a long-term mean, i.e., it *reverts* to the *mean*.

Mean Reverting Process

Uses

- Any variable that shows short-term deviations, but long-term return to the mean.
- Interest rates

Mean Reverting Process

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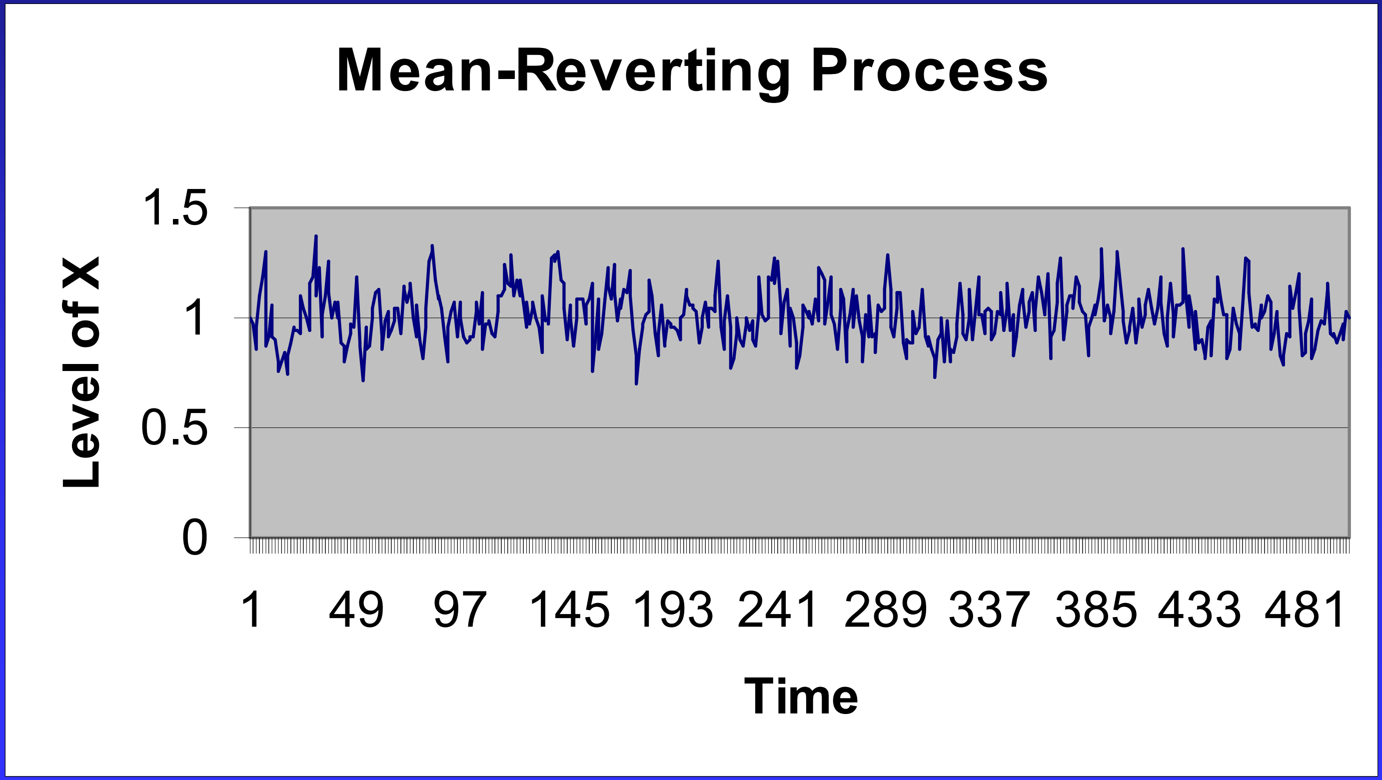
Formula

$$dX = \kappa(\mu - X)\,dt + \sigma X^{\gamma}\,dW$$

- dX = change in variable X
- κ = speed of adjustment parameter ($\kappa \geq 0$)
- μ = long-run mean ($\mu \geq 0$)
- γ = adjustment value (we will assume $\gamma = 1$)
- dt = change in time
- σ = volatility
- dW = Weiner process

Mean Reverting Process

Graph



Mean Reverting Process

Characteristics

- X returns to μ in the long-run
- The speed of adjustment parameter (κ) determines how quickly X returns to μ. The higher κ, the closer X stays to μ.
- If X begins at a positive value, it remains positive.
  - As X nears zero drift is positive and volatility goes to zero.

Summary

- We shall focus on:
  - Geometric Brownian motion for modeling stock prices, and
  - Mean-reverting processes for modeling interest rates.